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THE GAUGE EQUIVALENCE OF THE ZAKHAROV EQUATIONS AND (2+1)-DIMENSIONAL CONTINUOUS HEISENBERG FERROMAGNETIC MODELS[1]

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Abstract

The gauge equivalence between the (2+1)-dimensional Zakharov equation and (2+1)-dimensional integrable continuous Heisenberg ferromagnetic model is established. Also their integrable reductions are shown explicitly.

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The concepts of gauge equivalence between completely integrable equations plays important role in the theory of solitons[1,2]. In the (2+1)-dimensions such equivalence have been constructed recently for the Davey-Stewartson and Ishimori equations[3], for the some Myrzakulov and nonlinear Schrödinger type equations and so on[4-7]. In this Letter we wish find the gauge equivalent counterparts of the some (2+1)-dimensional integrable continuous Heisenberg ferromagnet models(the Myrzakulov-IX equation and its integrable reductions).

The Myrzakulov-IX(M-IX) equation (according to the notations of ref.[8]) looks like

$$iS_t + \frac{1}{2}[S, M_1 S] + A_2 S_x + A_1 S_y = 0 \tag{1a}$$

$$M_2 u = \frac{\alpha^2}{4i} tr(S[S_y, S_x]) \tag{1b}$$

where $\alpha, b, a=$ consts and

$$S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^{\pm} = S_1 \pm iS_2 \quad S^2 = I, \quad r^2 = \pm 1$$

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(b-a)\frac{\partial^2}{\partial x \partial y} + (a^2 - 2ab - b)\frac{\partial^2}{\partial x^2};$$

$$M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - \alpha(2a+1)\frac{\partial^2}{\partial x \partial y} + a(a+1)\frac{\partial^2}{\partial x^2},$$

$$A_1 = 2i\{(2ab + a + b)u_x - (2b+1)\alpha u_y\}$$

$$A_2 = 2i\{(2ab + a + b)u_y - \alpha^{-1}(2a^2 b + a^2 + 2ab + b)u_x\}.$$

These set of equations is integrable and admits the some integrable reductions: the Myrzakulov-VIII equation as b=0 and the Ishimori equation as $a = b = -\frac{1}{2}$[8]. In general we have the two integrable cases: the M-IXA equation as $\alpha^2 = 1$ and the M-IXB equation as $\alpha^2 = -1$. Equation(1) is the (2+1)-dimensional integrable generalisation of the Heisenberg ferromagnetic model $iS_t = \frac{1}{2}[S, S_{xx}]$ (the isotropic Landau-Lifshitz equation).

The Lax representation of the M-IX equation(1) is given by[8]

$$\alpha\Phi_y = \frac{1}{2}[S + (2a+1)I]\Phi_x \qquad (2a)$$

$$\Phi_t = \frac{i}{2}[S + (2b+1)I]\Phi_{xx} + \frac{i}{2}W\Phi_x \qquad (2b)$$

with

$$W_1 = W - W_2 = (2b+1)E + (2b - a + \frac{1}{2})SS_x + (2b+1)FS$$

$$W_2 = W - W_1 = FI + \frac{1}{2}S_x + ES + \alpha SS_y$$

$$E = -\frac{i}{2\alpha}u_x, \quad F = \frac{i}{2}(\frac{(2a+1)u_x}{\alpha} - 2u_y)$$

Let us now find the equation which is gauge equivalent to the M-IX equation(1). To this end, we consider the following tranformation

$$\Phi = g^{-1}\Psi \qquad (3)$$

where Φ is the matrix solution of linear problem(2), Ψ and g are a temporally unknown matrix functions. Substituting (3) into (2) we get

$$\alpha\Psi_y = \frac{1}{2}[gSg^{-1}+(2a+1)I]\Psi_x+[\alpha g_y-\frac{1}{2}gSg^{-1}g_x-\frac{1}{2}(2a+1)g_x]g^{-1}\Psi \qquad (4a)$$

$$\Psi_t = \frac{i}{2}[gSg^{-1}+(2b+1)I]\Psi_{xx}+g\{i[S+(2b+1)I](g^{-1})_x+\frac{i}{2}Wg^{-1}\}\Psi_x+$$

$$g\{g_tg^{-1} + \frac{i}{2}[S + (2b+1)I](g^{-1}_{xx} + \frac{i}{2}W(g^{-1})_x\}\Psi. \qquad (4b)$$

Now let us choose the unknown function g and S in the form

$$g = \begin{pmatrix} f_1(1+S_3) & f_1rS^- \\ f_2rS^+ & -f_2(1+S_3) \end{pmatrix}, \quad S = g^{-1}\sigma_3 g \qquad (5)$$

where f_j are satisfy the following equations

$$\alpha(\ln f_1)_y-(a+1)(\ln f_1)_x = \frac{(a+1)(S_{3x}+S_3S_{3x}+S^-_xS^+) - \alpha(S_{3y}+S_3S_{3y}+S^-_yS^+)}{2(1+S_3} \qquad (6a)$$

$$\alpha(\ln f_2)_y - a(\ln f_2)_x = \frac{a(S_{3x}+S_3S_{3x}+S_x^+S^-)-\alpha(S_{3y}+S_3S_{3y}+S_y^+S^-)}{2(1+S_3} \tag{6b}$$

It is follows from (4)-(6) that

$$\alpha g_y g^{-1} - B_1 g_x g^{-1} = B_0 \tag{7}$$

where

$$B_1 = \begin{pmatrix} a+1 & 0 \\ 0 & a \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}.$$

Here p, q are the some complex functions which are equal

$$q = \frac{f_1\{\alpha[S_{3y}S^- - S_y^-(1+S_3)] + (a+1)[S_{3x}S^- - S_x^-(1+S_3)]\}}{2f_2(1+S_3)} \tag{8a}$$

$$p = \frac{f_2\{a[S_{3x}S^+ - S_x^-(1+S_3)] + \alpha[S_{3y}S^+ - S_y^-(1+S_3)]\}}{2f_1(1+S_3)}. \tag{8b}$$

Hence we obtain

$$pq = \frac{1}{4}\{\alpha(2a+1)\mathbf{S}_x\mathbf{S}_y + i\alpha\mathbf{S}(\mathbf{S}_x\wedge\mathbf{S}_y) - a(a+1)\mathbf{S}_x^2 - \alpha^2\mathbf{S}_y^2\}. \tag{9}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is the three - dimensional spin(unit) vector. After these calculations equations (4) take the forms

$$\alpha\Psi_y = B_1\Psi_x + B_0\Psi, \tag{10a}$$

$$\Psi_t = iC_2\Psi_{xx} + C_1\Psi_x + C_0\Psi, \tag{10b}$$

with

$$C2 = \begin{pmatrix} b+1 & 0 \\ 0 & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & iq \\ ip & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

$$c_{12} = i(2b-a+1)q_x + i\alpha q_y, \quad c_{21} = i(a-2b)p_x - i\alpha p_y.$$

Here c_{jj} satisfy the following system of equations

$$(a+1)c_{11x} - \alpha c_{11y} = iqp_x + pc_{12} - qc_{21}, \; ac_{22x} - \alpha c_{22y} = ipq_x - pc_{12} + qc_{21} \tag{11}$$

The compatibility condition of equations(10) gives the following (2+1)-dimensional nonlinear Schrödinger type equation

$$iq_t + M_1 q + vq = 0, \tag{12a}$$

$$ip_t - M_1 p - vp = 0, \tag{12b}$$

$$M_2 v = -2M_1(pq), \tag{12c}$$

which is the Zakharov equation(ZE)[9], where $v = i(c_{11} - c_{22}), \quad p = r^2 q$. Thus we proved that the M-IX equation(1) is gauge equivalent to the (2+1)-dimensional ZE(12) and vice versa. In fact the above presented the gauge transformation is reversible.

It is interest note that the M-IX equation(1) admits the some integrable reductions. Let us now consider these particular integrable cases.

a) Let $b = 0$. Then equations(1) take the form

$$iS_t = \frac{1}{2}[S_{\xi\xi}, S] + iwS_\xi \tag{13a}$$

$$w_\eta = \frac{1}{4i} tr(S[S_\xi, S_\eta]) \tag{13b}$$

where

$$\xi = x + \frac{a+1}{\alpha} y, \quad \eta = -x - \frac{a}{\alpha} y, \quad w = u_\xi,$$

which is the M-VIII equation[8]. The gauge equivalent counterpart of the M-VIII equation(13) we obtain from(12) as $b = 0$

$$iq_t + q_{\xi\xi} + vq = 0, \tag{14a}$$

$$v_\eta = -2r^2(\bar{q}q)_\xi, \tag{14b}$$

which is the other Zakharov equation[9].

b) Now consider the case: $a = b = -\frac{1}{2}$. In this case equations(1) reduces to the well known Ishimori equation

$$iS_t + \frac{1}{2}[S, (\frac{1}{4}S_{xx} + \alpha^2 S_{yy})] + iu_y S_x + iu_x S_y = 0 \tag{15a}$$

$$\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4i}tr(S[S_y, S_x]) \qquad (15b)$$

The Ishimori equation (15) is of the great interest since it is the first example of the integrable spin systems on the plan. This equation is considered as a useful laboratory for experimenting with new theoretical tools ableto handle the specific featuresof soliton models of spin systems in (2+1)-dimensions. As well known equation(15) allows the rich classes of the topologically nontrivial and nonequivalent solutions(solitons, lumps, vortex, dromions and so on) which are classified by the topological charge

$$Q = \frac{1}{4\pi}\int\int_{-\inf}^{+\inf} dxdy\mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y) \qquad (16)$$

The gauge equivalent counterpart of the equation(60) is the Davey-Stewartson equation

$$iq_t + \frac{1}{4}q_{xx} + \alpha^2 q_{yy} + vq = 0 \qquad (17a)$$

$$\alpha^2 v_{yy} - \frac{1}{4}v_{xx} = -2\{\alpha^2(pq)_{yy} + \frac{1}{4}(pq)_{xx}\} \qquad (17b)$$

that follows from the ZE(12). This fact was for first time established in [3].

c) Finally let us consider the reduction: $a = -\frac{1}{2}$. Then (1) reduces to the M-XVIII equation[8]

$$iS_t + \frac{1}{2}[S, (\frac{1}{4}S_{xx} - \alpha(2b+1)S_{xy} + \alpha^2 S_{yy})] + A_{20}S_x + A_{10}S_y = 0 \qquad (18a)$$

$$\alpha^2 u_{yy} - \frac{1}{4}u_{xx} = \frac{\alpha^2}{4i}tr(S[S_y, S_x]) \qquad (18b)$$

where $A_{j0} = A_j$ as $a = -\frac{1}{2}$. The corresponding gauge equivalent equation obtain from(12) and looks like

$$iq_t + \frac{1}{4}q_{xx} - \alpha(2b+1)q_{xy} + \alpha^2 q_{yy} + vq = 0 \qquad (19a)$$

$$\alpha^2 v_{yy} - \frac{1}{4}v_{xx} = -2\{\alpha^2(pq)_{yy} - \alpha(2b+1)(pq)_{xy} + \frac{1}{4}(pq)_{xx}\} \qquad (19b)$$

Note that the Lax representations of equations(13), (15) and (18) we can get from (2) as $b = 0, a = b = -\frac{1}{2}$ and $a = -\frac{1}{2}$ respectively.

In summary, we constructed the gauge equivalent equation to the M-IX equation which is the Zakharov equation. Also the integrable reductions of the M-IX equation and their gauge equivalent counterparts are presented.

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